Optimal Energy Assignment for Frequency Selective Fading Channels

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Abstract — **This paper describes a method to minimize the total energy necessary to communicate data over a frequency selective fading channel using multicarrier data transmission. The energy is minimized given a desired average bit error rate at a fixed gross bit rate. For a number of configurations the optimal energy distribution is calculated analytically. For other configurations we present a generic numerical approach. To evaluate the performance, a comparison is made with two other commonly used energy assignment schemes. The advantage of this scheme is most apparent when the channel suffers from deep fades, as is the case in mobile wireless communications.**

I. Introduction

In a multipath channel, data transmission suffers from frequency selective fading. At high symbol rates, this results in severe inter symbol interference (ISI). An approach to overcome this problem is to divide the available bandwidth into many independent narrow sub-bands and assign each sub-band to a so-called subchannel [1].

Instead of transmitting the symbols serially over a single channel, the symbols are transmitted in parallel over the subchannels. As a result, the symbol rate per subchannel is lowered, effectively reducing the inter symbol interference, while maintaining the overall symbol rate. Additional measures, such as the use of a cyclic prefix [2], effectively annul the remaining ISI.

Orthogonal frequency division multiplexing (OFDM), also known as multicarrier modulation (MCM) or discrete multitone (DMT), is a popular transmission scheme, which applies such a technique. It has been proposed and/or used for several international standards such as digital audio broadcasting (DAB) [3], universal mobile telecommunications system (UMTS) [4] and asymmetric digital subscriber lines (ADSL) [5].

Multicarrier systems allow different energy levels to be assigned to each subchannel. This paper describes a method that exploits this property to minimize the total energy necessary given a desired average bit error rate. The gross bit rate is determined by the number of subchannels and the symbol mapping scheme per subchannel.

II. Problem description

A. Channel model

The channel model under consideration consists of N parallel narrowband subchannels. Since the ISI is effectively canceled, the subchannels do not suffer from frequency selective fading. We model the remaining frequency non-selective fading (or flat fading) with a power attenuation factor C_i for every subchannel i. This factor is a piecewise constant approximation (see Fig. 1) of the channel transfer function $H(f)$ and is given by

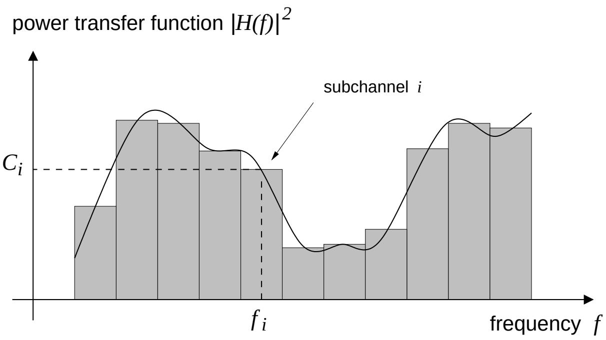


Fig. 1. Piecewise constant approximation of the power transfer function of the channel.

$$C_i = |H(f_i)|^2, \tag{1}$$

where f_i is the central frequency of subchannel i.

This approximation is valid when the subchannels are narrowband. A subchannel is considered to be narrow band if the bandwidth of the subchannel B_i is sufficiently smaller than the coherence bandwidth B_c of the channel

$$B_i \ll B_c. \tag{2}$$

As a rule of thumb the coherence bandwidth can be approximated using [6]

$$B_c \approx \frac{1}{T_m}, \tag{3}$$

where T_m is the multipath spread of the channel.

In addition to flat fading, the subchannels also suffer from additive white Gaussian noise (AWGN) which limits their transmission capacity. The noise power N_0 is assumed to be constant and equal for all subchannels. For the sake of clarity, we neglect inter cell interference and inter carrier interference. However, with some modifications to the model presented here, both types of interference can be incorporated as well.

B. System model

On each subchannel a modulated signal is transmitted. On every subchannel i an average bit energy E_{b_i} is used to transmit the information. The probability of a bit error on the i^{th} subchannel is a function of the average bit energy, the power attenuation factor C_i and the average noise power N_0

$$P_{e_i} = F\left(E_{b_i}, C_i, N_0\right). \tag{4}$$

In general F is a strictly decreasing function with respect to E_{b_i}. The precise form depends on the applied symbol mapping scheme.

Throughout this paper, we assume a uniform modulation scheme for each subchannel. This assumption is made only for analytical convenience and does not limit the validity of the model (see section III-B).

The total energy transmitted over the channel E_b is the sum of the energies transmitted over all subchannels, i.e.

$$E_b = \sum_1^N E_{b_i}, \tag{5}$$

where N is the number of subchannels. The average bit error rate is given by the mean of the bit error probabilities per subchannel

$$P_e = \frac{1}{N}\sum_1^N P_{e_i}. \tag{6}$$

The total number of bits B transmitted, depends on the number of bits per symbol b_i which, still assuming a uniform symbol mapping scheme for all subchannels, is given by

$$B = \sum_1^N b_i = N \cdot b, \tag{7}$$

where b is the number of bits per subchannel.

C. Optimization problem

With the definitions above we formulate the following optimization problem:

Find the average bit energies $E_{b_i} \forall\, i = 1 \ldots N$, with N the number of subchannels, such that the total energy E_b is minimized

$$\min\left[E_b = \sum_1^N E_{b_i}\right],$$

while the average bit error probability P_e does not exceed the desired error rate R_e

$$P_e = \frac{1}{N}\sum_1^N P_{e_i} \leq R_e.$$

III. Solution

Since the individual error probability functions per subchannel P_{e_i} are strictly decreasing with respect to the average bit energy per subchannel E_{b_i}, the overall average bit error probability P_e is also strictly decreasing. An optimum is reached when the partial derivatives with respect to the average bit energy per subchannel of the average bit error probability are equal, that is

$$\frac{\partial P_e}{\partial E_{b_i}} = \frac{\partial P_e}{\partial E_{b_j}}. \tag{8}$$

The solution to (8) gives the relative energy distribution. To find the absolute energy per subchannel the additional boundary condition has to be solved, i.e.

$$P_e = \frac{1}{N}\sum_1^N P_{e_i} = R_e, \tag{9}$$

where R_e is the desired average bit error rate. Solving (8) and (9) simultaneously gives the optimal energy distribution to achieve the desired bit error rate at the lowest total energy cost.

A. QPSK optimization

In case of QPSK symbol modulation we are able to derive a closed formula for the relative energy distribution. The error probability per subchannel P_{e_i} for QPSK symbol modulation is given by [7]

$$P_{e_i} = Q\left(\sqrt{2\frac{C_i E_{b_i}}{N_0}}\right). \tag{10}$$

Q is defined, as usual, by

$$Q(z) = \frac{1}{\sqrt{2\pi}}\int_z^\infty e^{-\frac{1}{2}\lambda^2} d\lambda. \tag{11}$$

The partial derivative with respect to the average energy per bit then boils down to

$$\frac{\partial P_e}{\partial E_{b_i}} = -\frac{1}{2}\sqrt{\frac{a_i}{\pi E_{b_i}}}\, e^{-a_i E_{b_i}}, \tag{12}$$

with $a_i = \frac{C_i}{N_0}$. Substitution of (12) in (8) gives

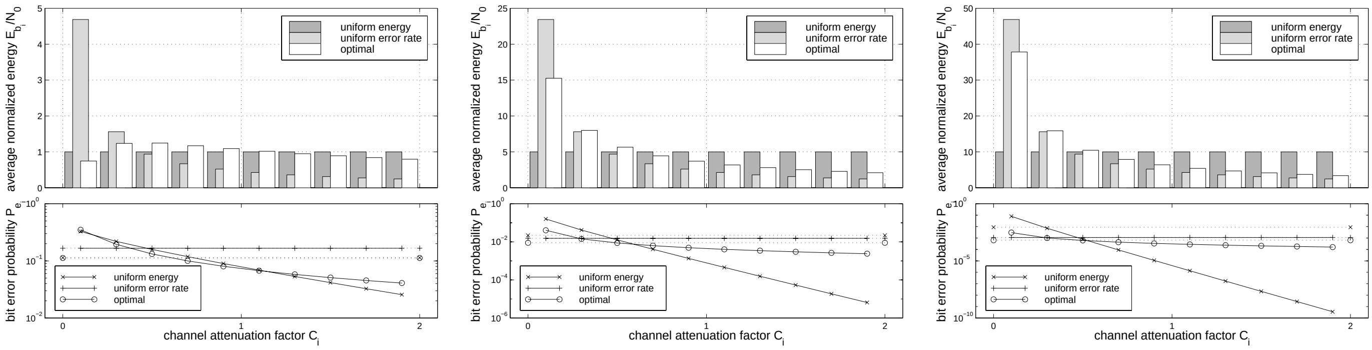
$$\sqrt{\frac{a_i}{E_{b_i}}}\, e^{-a_i E_{b_i}} = \sqrt{\frac{a_j}{E_{b_j}}}\, e^{-a_j E_{b_j}}. \tag{13}$$

Squaring and rearrangement leads to

$$a_i E_{b_j}\, e^{2a_j E_{b_j}} = a_j E_{b_i}\, e^{2a_i E_{b_i}} \tag{14}$$

and finally solving (14) for E_{b_j}, substituting $a_i = \frac{C_i}{N_0}$ and assuming a unity reference attenuation factor $C_i = 1$ gives the optimum normalized energy

$$\frac{\widehat{E}_{b_j}}{N_0} = \frac{1}{2C_j} W\left(2C_j^2 \frac{E_{b_i}}{N_0}\, e^{2\frac{E_{b_i}}{N_0}}\right). \tag{15}$$



(a) $E_b/N_0 = 10$ (b) $E_b/N_0 = 50$ (c) $E_b/N_0 = 100$

Fig. 2. Comparison of the energy distributions (E_b/N_0 - total normalized energy).

$W(x)$ is Lambert's W function, which obeys

$$W(x)\, e^{W(x)} = x. \tag{16}$$

Equation (15) gives the relative energy distribution over the subchannels, the absolute values can be found by an iterative approximation of (9) using for example Newton-Raphson. As the bit error probability P_{e_i} is a strictly decreasing function of E_{b_i}, convergence is guaranteed.

B. QAM-16 optimization

In case of QAM-16 symbol modulation we derive a numerical procedure to calculate the optimal energy distribution. The bit error probability per subchannel P_{e_i} for QAM-16 symbol modulation is given by [8]

$$P_{e_i} = \frac{3}{4} Q\left(\sqrt{\frac{1}{5}\frac{C_i E_{b_i}}{N_0}}\right) + \frac{1}{4} Q\left(\sqrt{\frac{9}{5}\frac{C_i E_{b_i}}{N_0}}\right). \tag{17}$$

Setting the partial derivatives equal and solving the resulting equations however, does not yield a simple analytical solution. Therefore the solution is calculated numerically using the following procedure:

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1. *assign* $E_{b_i} \forall\, i = 1 \ldots N$ *such that* $P_{e_i} = R_e$
2. *calculate* $\left|\frac{\partial P_e}{\partial E_{b_i}}\right|_{E_{b_i}} \forall\, i = 1 \ldots N$
3. *while* $\left|\frac{\partial P_e}{\partial E_{b_i}}\right|_{E_{b_i}} \neq \left|\frac{\partial P_e}{\partial E_{b_j}}\right|_{E_{b_j}} \forall\, i = 1 \ldots N, j = 1 \ldots N$
   (a) *select the subchannels with the largest and smallest partial derivatives* $\left|\frac{\partial P_e}{\partial E_{b_i}}\right|_{E_{b_i}}$
   (b) *increase the energy for the subchannel with the largest partial derivative*
   (c) *find the new* E_{b_i} *for the subchannel with the smallest partial derivative while satisfying* $P_e = R_e$
   (d) *calculate the new partial derivatives*

The procedure above can be applied for any modulation scheme with a bit error probability P_{e_i} per subchannel of the form

$$P_{e_i} = \sum_{1}^{K} \alpha_k Q\left(\sqrt{\gamma_k \frac{C_i E_{b_i}}{N_0}}\right), \tag{18}$$

where K is an arbitrary integer and α_k and γ_k are modulation dependent constants. Application for non-uniform symbol modulation schemes is possible. To guarantee convergence a necessary and sufficient condition is a strictly decreasing bit error probability per subchannel P_{e_i} with respect to E_{b_i}.

IV. Comparison

In order to validate and demonstrate the effectiveness of the proposed optimal energy assignment scheme, we compare our method with two other widely used energy assignment schemes. The first is the traditional approach to assign all subchannels the same energy per bit (uniform energy). The second scheme assigns the energies in such a way that all subchannels have the same error rate (uniform error rate). In all cases we use QPSK for the symbol modulation.

Obviously the uniform energy assignment scheme does not take the channel conditions into account when assigning energy to each subchannel, whereas the uniform error rate assignment scheme does. However the latter only locally optimizes the energy for each subchannel. The proposed optimal method globally optimizes the energy assigned to each subchannel, making maximum use of the channel characteristics and the available energy.

A. Energy distribution

Fig. 2 shows (upper graphs) the energy distributions for each of the assignment methods for increasing energy levels. In the lower graphs the bit error probabilities are plotted for each subchannel, together with the resulting average error probability (dotted lines).

From the upper graphs, we observe that the optimal scheme assigns relatively less energy to the (poor) subchannels with low attenuation factors in comparison with the uniform error rate scheme and at the same time, it assigns more energy to the (good) subchannels with higher attenuation factors. The higher error rate for poor subchannels is compensated for by a lower error rate for good subchannels, resulting in a decrease of the average error rate.

Comparing the error probability curves in the lower graphs, it is observed that for low energy levels, the optimal scheme compares to the uniform energy scheme. For higher levels, it acts more like the uniform error rate assignment scheme.

B. Performance

The three assignment schemes are compared by simulation, using a Ricean fading channel model [6]. For different numbers of subchannels, a large number of channel responses is generated representing varying channel conditions. This is achieved by changing the Ricean K-factor, which represents the severity of multipath effects in the channel. The lower the K-factor, the more the multipath effects dominate. In case the Ricean K-factor equals zero, the Ricean fading channel model turns into a Rayleigh fading channel model.

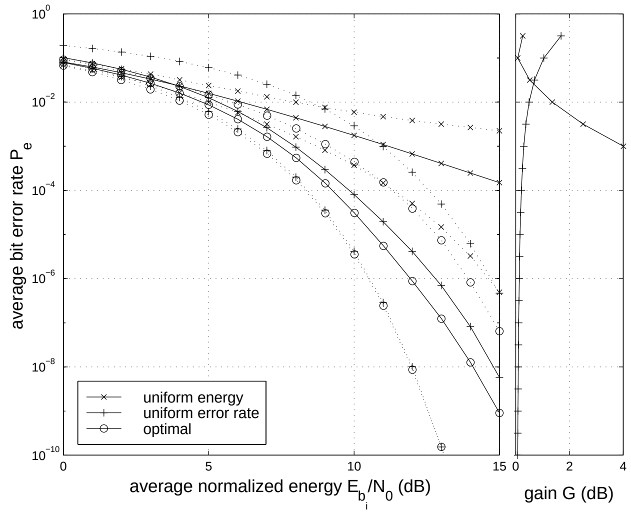
The performance of the schemes is compared by plotting the average bit error rate P_e against the average normalized energy per subchannel E_{b_i}/N_0 (see Fig. 3, left-hand graphs, solid curves). Next to the average, the performance with the best and worst channel conditions are shown (dashed curves). Additionally the energy gain G of the optimal scheme over the other two schemes is plotted (Fig. 3, right-hand graphs).

In all cases the simulations show that the scheme proposed in this paper has the best performance. The improvement is most significant (over 4 dB) with respect to the uniform energy distribution. The gain is moderate (in the order of half a dB) with respect to the uniform error rate distribution scheme. The most energy is gained (compare Fig. 3, right-hand graphs) when the channel conditions are worst, i.e. when the channel suffers from deep fades ($K \approx 0$).

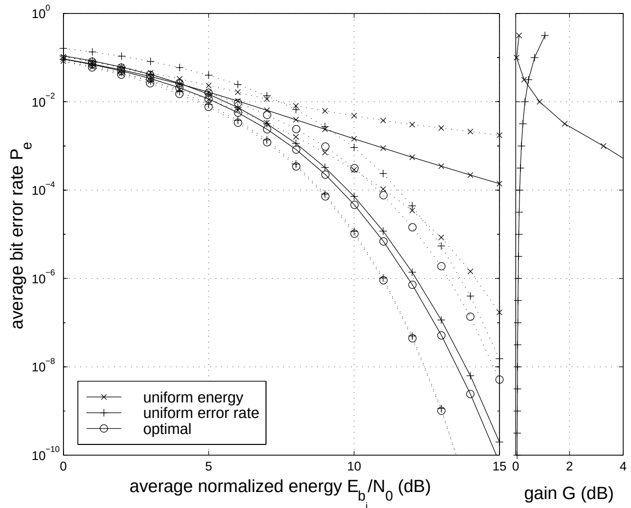
V. Conclusions

In this paper an optimal energy assignment scheme is proposed to minimize the total energy necessary to achieve a desired average bit error rate over frequency selective fading channels. For analytical convenience this paper concentrates on uniform symbol modulation for all subchannels. In addition to the analytical method, we also presented a generic numerical procedure, which also applies to non-uniform symbol modulation.

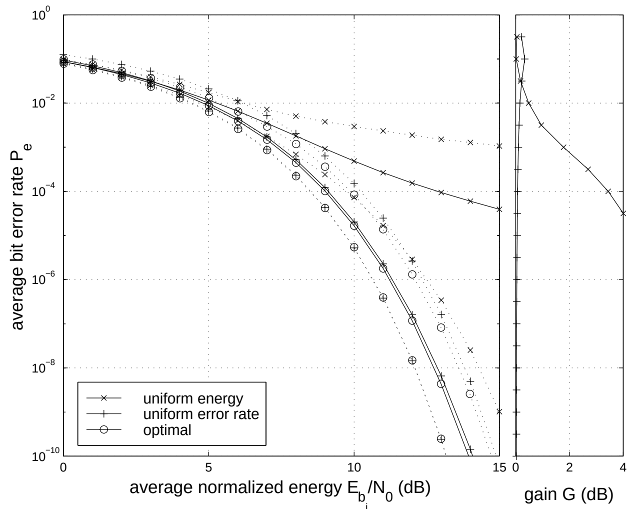
The performance of the proposed scheme is compared with two other commonly used assignment schemes. In all cases the optimal scheme shows the best performance. The most significant performance improvement occurs when the channel suffers from deep fades, as is the case with mobile wireless communications.



(a) $K = 0$ (Rayleigh fading)



(b) $K = 3$ dB



(c) $K = 6$ dB

Fig. 3. Performance comparison of distribution schemes over a Ricean fading channel (K - Ricean K-factor, number of subchannels $N = 128$, number of simulations $n = 100$).

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