EFFECT OF SMOOTH NON-LINEAR DISTORTION ON OFDM BER

Chris van den Bos, Michiel H.L. Kouwenhoven, and Wouter A. Serdijn

Delft University of Technology, Faculty of Information Technology and Systems
Mekelweg 4, 2628 CD Delft, Netherlands
e-mail: c.vandenbos@its.tudelft.nl

ABSTRACT

Non-linear distortion of OFDM signals can significantly increase the receiver BER. Currently, distortion analyses rely on simulation, not yielding insight into the problem, or apply to specific non-linearities only. In this paper, general analytic results are presented on the errors resulting from distortion. The results are applicable to smooth non-linear distortion of an OFDM signal, which is the most common distortion. Simple analytical expressions are derived which allows a designer to determine the BER before performing simulations. Further, it is shown that the error on each OFDM subcarrier is approximately equally large.

1. INTRODUCTION

OFDM (Orthogonal Frequency Division Multiplexing) is a modulation scheme that is capable of overcoming intersymbol interference (ISI) on frequency selective channels in a very efficient way. A digital information stream is divided into multiple information streams. This lowers the bitrate on each of these streams. The datastreams are modulated and mapped on orthogonal carriers (called "subcarriers"). Thus, many low-bitrate signals are transmitted, instead of one high bitrate signal. The low bitrate signals hardly suffer from ISI, and because of the orthogonality of the subcarriers, it is possible to demodulate the received signal without crosstalk between the information on the subcarriers. The performance of OFDM can approach the theoretical maximum for a given radio channel, [1].

A disadvantage of OFDM is its sensitivity to non-linear distortion, which causes crosstalk between subcarriers. In a matched-filter receiver, the crosstalk can dramatically increase the BER. As any real communication system will contain non-linearities, it is important to determine the resulting signal deterioration. Especially the antenna output amplifier of a transmitter can cause significant non-linear distortion.

So far, the distortion problem for matched filter receivers has been treated in two ways: either by simulation, [2, 3], or by mathematical analysis, [4]. The simulation results do not yield much insight into the problem. The reported analytical results in literature apply to specific clipping non-linearities only and cannot be used for design purposes. The effect of smooth non-linear distortion on matched filter reception of an OFDM signal is more relevant, as it is the most common type of distortion.

In this paper, analytical results are presented which apply to distortion of bandpass OFDM signals due to arbitrary memoryless, smooth non-linear transfers. They enable a designer to estimate the influence of non-linearity before performing simulations.

The organization of this paper is as follows. In section 2, the model for non-linear transfer is discussed and the relations with
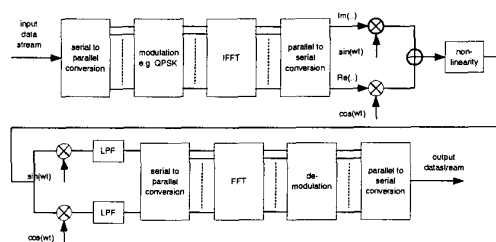

Figure 1: Practical OFDM system implementation

common electronic performance measures such as IP2 and IP3 are given. In section 3, the SDR (signal-to-distortion ratio) due to non-linearity is calculated. Subsequently the BER after detection by a matched filter receiver is calculated. In section 4, an example for QPSK-OFDM is treated both by analysis and simulation, to show the validity of the distortion model. In section 5, conclusions are presented.

2. SMOOTH NON-LINEARITY MODEL

Non-linear distortion of OFDM signals causes the loss of orthogonality of subcarriers, and thereby inter-carrier interference. The result is an increase of the symbol-error rate when the OFDM signal is detected by a matched-filter receiver.

In figure 1, a practical OFDM system including non-linearity is shown. In this system, an IFFT and FFT are used for modulation and demodulation. Essentially, it is a discrete-time system. A continuous-time model is used throughout this paper, as the bandpass OFDM signal is continuous.

The analysis in this paper is restricted to smooth non-linear transfers. These are likely to be dominant, and mainly originate in the OFDM transmitter output amplifier. A suitable approximation of such an amplifier is a memoryless non-linearity.

The non-linear distortion performance of amplifiers is often expressed in terms of intercept points (IP2, IP3), or intermodulation points (IM2, IM3). The parameters of the model have to be related to these measures. As intercept points and intermodulation points yield the same information for a memoryless non-linearity, only the relation between intercept points and the model parameters is given.

The output signal $y(t) = f[x(t)]$ of a memoryless non-linearity at a certain time instant only depends on the input signal at that same time instant. As $f(x)$ is smoothly non-linear, it can be accu-

rately represented by a few (K) terms of a Taylor polynomial:

$$f(x) \approx \sum_{k=0}^{K-1} a_k x^k \quad \text{where} \quad a_k = \frac{1}{k!} \frac{\partial^k f(x)}{\partial x^k}\Bigg|_{x=0} \quad (1)$$

According to [5], intercept points and the model parameters are related by

$$\text{IP2} = \frac{2a_1}{a_2} \quad \text{and} \quad \text{IP3} = \sqrt{\frac{4a_1}{a_3}} \quad (2)$$

It is important to note that even-order distortion (k even) is mapped to bands that are far from the original OFDM signal passband. It is assumed that these are £ltered out and do not in¤uence the BER. Therefore, only odd-order distortion is of importance.

3. SIGNAL DISTORTION ANALYSIS

In this section, the effect of distortion on the BER of an OFDM signal is determined. First, the spectrum of a distorted OFDM signal is examined. It is used to determine the detection error for matched £lter detection. After this, the BER can be determined.

The complex lowpass OFDM signal is described by [6]

$$s_l(t) = \sum_{n=0}^{N-1} d_n(t) e^{j2\pi nt/T} \quad (3)$$

d_n are complex-valued digital symbols that are constant during T seconds, N represents the number of complex subcarriers, and $e^{j2\pi nt/T}$ represent complex-valued orthogonal subcarriers. N is typically larger than 100. The d_n carry information and are therefore random variables. They are assumed to be independent, of zero mean, and of identical distribution, so the same modulation on each subcarrier. In practice, $s_l(t)$ is often formed by performing an IFFT on the symbols d_n.

The power spectrum of an OFDM bandpass signal is approximately rectangular, and occupies a bandwidth of N/T Hz. The amplitude distribution of an OFDM signal is nearly gaussian. This can be understood by noting that an OFDM signal consists of a sum of a large number of independent, identically distributed signals, and applying the Central Limit Theorem. The fact that the amplitude distribution is gaussian is used for determining the spectrum of the distorted OFDM signal.

In the remainder of this paper, only third-order distortion is taken into account for BER calculation, because for smooth nonlinearities third-order distortion is often dominant. Even if it is not, inclusion of third-order distortion only will give a good estimate of the BER to be expected. If necessary, the procedure that is followed can also be used for calculating the in¤uence of higher-order distortion.

3.1. Distortion spectral analysis

In this section, the spectrum of the third power of the OFDM signal only is calculated, equation (1). The result is used to calculate the distortion variance after matched £ltering. After matched-£lter detection, the OFDM signal and the distortion are nearly uncorrelated, which can be explained as follows. Due to the distortion, orthogonality between subcarriers is lost, so each OFDM subcarrier symbol will contribute to other subcarrier symbols and itself.

The number of contributions due to other subcarriers will outnumber the contribution of any subcarrier to itself. Thus, the correlation between OFDM signal and distortion is nearly zero and need therefore not be calculated.

The spectrum is calculated through the autocorrelation function which is

$$R_{x^3}(\tau) = R_{x^3}(t_2 - t_1) = E(x_{t_1}^3 x_{t_2}^3) \quad (4)$$

This expression is evaluated using a theorem for stochastic processes that are gaussian in distribution, [7].

Theorem 1 *If $n_1, \ldots, n_{2k}$ denote an even number of zero-mean Gaussian random variables, not necessarily independent, then*

$$E(n_1 \ldots n_{2k}) = \sum_{\text{all pairs } i \neq j} \prod^{k} E(n_i n_j) \quad (5)$$

As an example, suppose a gaussian signal $x(t)$ is squared, so $y(t) = x^2(t)$. The autocorrelation function of y is given by

$$R(t_1, t_2) = E(y_{t_1} y_{t_2}) = E(x_{t_1} x_{t_1} x_{t_2} x_{t_2}) \quad (6)$$

From the theorem, for $k = 2$, it follows

$$
\begin{aligned}
E(n_1 n_2 n_3 n_4) &= E(n_1 n_2)E(n_3 n_4) & (7) \\
&+ E(n_1 n_3)E(n_2 n_4) & (8) \\
&+ E(n_1 n_4)E(n_2 n_3) & (9)
\end{aligned}
$$

Substituting $n_1 = x_{t_1}$ and $n_2 = n_3 = n_4 = x_{t_2}$ it is found that

$$E(x_{t_1} x_{t_1} x_{t_2} x_{t_2}) = E(x_{t_1}^2)E(x_{t_2}^2) + 2E^2(x_{t_1} x_{t_2}) \quad (10)$$

so the desired correlation function is

$$R_y(\tau) = \sigma_x^4 + 2R_x^2(\tau) \quad (11)$$

where $\tau = t_1 - t_2$. A similar calculation can be performed for $E(x_{t_1}^3 x_{t_2}^3)$, yielding

$$R_{x^3}(\tau) = 6R_x^3(\tau) + 9\sigma_x^4 R_x(\tau) \quad (12)$$

which is the desired autocorrelation function.

The autocorrelation function and the power spectral density are related by

$$
\begin{aligned}
S_{x^3}(f) &= \int_{-\infty}^{\infty} R_{x^3}(\tau) e^{-j2\pi f\tau} d\tau \\
&= 6S_x(f) * S_x(f) * S_x(f) + 9\sigma_x^4 S_x(f) \quad (13)
\end{aligned}
$$

To evaluate the convolution, knowledge of the original OFDM spectrum is required. It is assumed that the OFDM spectrum is rectangular, with power σ_x^2 and nonzero for $f_0 - B < |f| < f_0 + B$. This is a good approximation of the real spectrum.

Using an expression for threefold convolution of a rectangular spectrum, adopted from [8], it is found that the bandpass spectrum of the distortion that is uncorrelated with the OFDM signal after detection is

$$
S_{x^3}(f) = \frac{9\sigma_x^6}{4B} a_3^2 \begin{cases} 1 & |f \pm f_0| < B \\ 0 & |f \pm f_0| > B \end{cases} +
$$

$$
\frac{3\sigma_x^6}{32B^3} a_3^2 \begin{cases} 3B^2 - (f \pm f_0)^2 & |f \pm f_0| < B \\ \frac{1}{2}(3B - |f \pm f_0|)^2 & B < |f \pm f_0| < 3B \quad (14) \\ 0 & |f \pm f_0| > 3B \end{cases}
$$

which is used to calculate the SDR.

3.2. Influence of distortion on correlation reception

In this section, the influence of the distortion on the BER is investigated. In many practical receivers, matched filter detectors are used. In case of Gaussian noise, a matched filter detector is optimal.

However, the distribution of the distortion is not gaussian, and therefore the distribution of the distortion after matched filtering cannot be assumed to be gaussian. The exact calculation of the distribution is a difficult problem.

Fortunately, a result by Papoulis, [9] states that a random process that is narrowband filtered has an output amplitude distribution that approaches a gaussian distribution. As the OFDM matched filters are narrowband with respect to the signal and the distortion, it can be concluded that the distortion must be approximately gaussian. Further, it can be shown that the distortion on $Re(d_n)$ and $Im(d_n)$ of a subcarrier is uncorrelated.

Knowing the distortion at the output of the matched filter is gaussian, it suffices to calculate the mean and the variance of the distortion on each subcarrier to completely characterize the stochastic process. As only the third-order distortion contributes to in-band distortion components, an amplifier model with only third-order distortion is used:

$$y = a_1 x + a_3 x^3 \tag{15}$$

The relation between input and output spectrum of a filter is given by

$$S_y(f) = S_x(f)|H(f)|^2 \tag{16}$$

where $H(f)$ is the transfer function of the filter.

When $H(f)$ is a matched filter for OFDM, it is a sinc-function with the peak at the frequency of the desired subcarrier, so for subcarrier n:

$$|H(f)|^2 = \{\mathrm{sinc}[\pi(n - (f - f_0)T)]\}^2 \tag{17}$$

where $\mathrm{sinc}(x) \equiv \frac{\sin(x)}{x}$. The distortion spectrum is given by equation (14). To determine the distortion power, the variance of $S_x(f)|H(f)|^2$ has been calculated for each subcarrier. Two calculations have been performed: one including the distortion outside the principal OFDM bandwidth N/T, and one excluding this distortion. There was no significant difference between the two graphs. Therefore it is not useful to filter out the distortion outside the principal OFDM band.

It found that the variance is nearly the same for each subcarrier, so the distortion is (nearly) evenly spread over all subcarriers, see also figure 2. The bit error rates on all subcarriers are therefore nearly equal.

The input distortion variance is found by integrating the distortion power spectrum. Straightforward calculation yields an in-band distortion power of

$$10a_3^2\sigma_x^6 \tag{18}$$

The signal power is $a_1^2\sigma_x^2$, so the SDR is given by

$$SDR = \left(\frac{a_1}{a_3}\right)^2 \frac{1}{10\sigma_x^4} \tag{19}$$

where SDR is the signal to uncorrelated distortion ratio. The same SDR is found at the output of a matched filter, because the spectra of OFDM signal and uncorrelated distortion have the same shape in the frequency region of interest.

a_3	IP3(dB)	distortion power (calculated)	distortion power (simulated)
-1	6	10.0	11.8
-0.1	16	0.10	0.119
-0.01	26	$1.0 \cdot 10^{-3}$	$1.21 \cdot 10^{-3}$

Table 1: Calculated and simulated distortion power per subcarrier; $a_1 = 1$.

4. EXAMPLE

In this section, an example is discussed. A bandpass QPSK-OFDM signal passes through a non-linear amplifier and is detected by a matched filter detector. Subsequently, the distortion variance is determined by means of simulation and calculation, and a comparison is made. Finally the BER is determined.

If the OFDM signal power is given by σ_x^2 and the amplifier transfer is $y = x + a_3 x^3$, the SDR is given by

$$SDR = \frac{1}{10a_3^2\sigma_x^4} \tag{20}$$

The SDR at the output of the matched filter is the same as the SDR at the input. The energy of a QPSK symbol E_s is set to 1, so the QPSK power per subcarrier is also 1. Thus, the distortion power per subcarrier should be $\sigma_d^2 = \frac{P_{QPSK}}{SDR} = \frac{1}{SDR}$. Thus, we have that

$$\sigma_d^2 = 10a_3^2\sigma_x^4 \tag{21}$$

For $\sigma_x^2 = 1$, the calculation has been performed for several values of a_3. The results are summarized in table 1 for easy comparison to simulation results. IP3 has also been included; as no units have been used, IP3 is denoted in dB (no reference power).

The simulations have been performed using MATLAB. Essentially, the system depicted in figure 1 has been implemented. To circumvent up-and downconversion to an RF frequency, an equivalent low-pass representation of the non-linearity has been used, which is allowed for in-band distortion.

The OFDM modulator performs an IFFT on a complex input vector. The input vector elements represent the QPSK symbols. The output vector of the IFFT is processed by the non-linearity. The output vector is demodulated by an FFT.

The QPSK symbols in the original and the demodulated vector are compared. The difference is squared and stored for each symbol separately. The procedure is repeated several times, and the results are averaged. Thus, for each subcarrier, the distortion variance can be depicted. An example is shown in figure 2.

The BER vs a_3 curve is shown in figure 3. It can be seen that there is a sharp drop in the simulated BER for $a_3 \approx -0.11$. Previously, only the effect of distortion and gaussian noise together has been studied [2, 4]. When noise and distortion are present, they both contribute to errors, thereby masking the influence of the distortion alone. Therefore the influence of distortion alone has been investigated in this paper. If necessary, the BER calculations in this paper can be extended by using the signal-to-noise-and-distortion ratio (SNDR) instead of using the SDR [4]. The SNDR is given by

$$\frac{1}{SNDR} = \frac{1}{SNR} + \frac{1}{SDR} \tag{22}$$

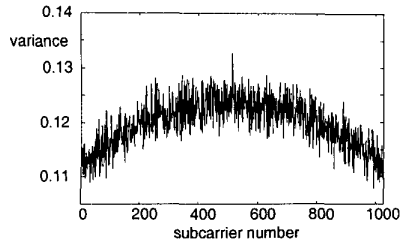where SNR is the signal-to-noise ratio without distortion.

Figure 2: The variance per subcarrier from simulation: $N = 1024$, $a_3 = -0.1$, $\sigma_x^4 = 1$, average for 1000 OFDM symbols.
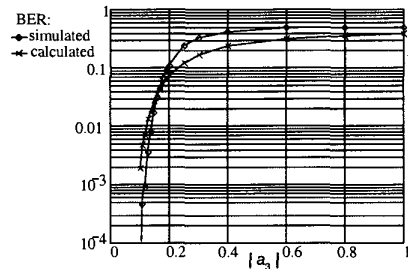

Figure 3: The BER of a QPSK-OFDM signal as a function of third-order distortion

From the graph it can be seen that the simple expression for the distortion can well be used to estimate the minimum allowable IP3.

5. CONCLUSIONS

In this paper, the effect of memoryless, smooth non-linear distortion on bandpass OFDM signals has been investigated. In particular, the OFDM power spectrum and the bit error rate after linear detection have been analyzed. The method employed is applicable to any memoryless, smooth non-linear distortion, so it is not restricted to special cases only.

Only the effect on third-order distortion has been taken into account. Inclusion of third-order distortion is suf£cient for a good £rst approximation. If needed, the analysis can readily be extended to include higher-order distortion as well.

The results are simple analytical expressions, which can be evaluated by manual calculation. This is particularly interesting to designers who would like to predict system behaviour before performing any simulations. Comparison of analytical and simulation results shows that the simple expressions can be used to estimate the minimum allowable IP3.

The signal deterioration has been examined for each separate subcarrier, which revealed that the error on each subcarrier is approximately equal. This means that as far as distortion is concerned, it does not make sense to distribute data over subcarriers in a particular way to increase the overall performance.

6. ACKNOWLEDGEMENT

7. REFERENCES

[1] J. A. C. Bingham, "Multicarrier modulation: an idea whose time has come," *IEEE Communications Magazine*, vol. 28, pp. 5–14, May , 1990.

[2] S. Merchán, A. G. Armada, and J. García, "OFDM performance in amplifer nonlinearity," *IEEE Transactions on Broadcasting*, vol. 44, pp. 106–114, March , 1998.

[3] X. Li and L. J. Cimini, "Effects of clipping and distortion on the performance of OFDM," *IEEE Communications Letters*, vol. 2, pp. 131–133, May , 1998.

[4] Q. Shi, "OFDM in bandpass nonlinearity," *IEEE Transactions on Consumer Electronics*, vol. 42, pp. 253–258, August , 1996.

[5] B. Gilbert, "Design considerations for bjt active mixers," in *Low-Power HF Microelectronics: A Uni£ed Approach* (G. A. S. Machado, ed.), IEE Circuits and Systems Series 8, London, UK: The Institution of Electrical Engineers, 1996.

[6] O. Edfors, M. Sandell, J. van de Beek, D. Landstrom, and F. Sjoberg, "An introduction to orthogonal frequency-division multiplexing." available from http://www.sm.luth.se/csee/sp/publications/, 1996.

[7] D. Middleton, *An Introduction to Statistical Communication Theory*. New York: McGraw-Hill Book Company, 1960.

[8] B. F. McGuf£n, "The effect of bandpass limiting on a signal in wideband noise," in *proceedings MILCOM*, (New York, USA), pp. 206–210, 1992.

[9] A. Papoulis, "Narrow-band systems and gaussianity," *IEEE Transactions on Information Theory*, vol. IT-18, pp. 20–27, January , 1972.