Effect of Smooth Nonlinear Distortion on OFDM Symbol Error Rate

Chris van den Bos, Michiel H. L. Kouwenhoven, *Member, IEEE*, and Wouter A. Serdijn, *Associate Member, IEEE*

***Abstract*—Nonlinear distortion of OFDM signals can significantly increase the receiver symbol error rate (SER). Currently, distortion analyses rely on simulation, not yielding insight into the problem, or apply to specific nonlinearities only. In this letter, general analytic results are presented on the errors resulting from distortion. The results are applicable to smooth nonlinear distortion of an OFDM signal, which is the most common distortion. Simple analytical expressions are derived which allows a designer to determine the SER before performing simulations. Further, it is shown that the error on each OFDM subcarrier is approximately equally large.**

I. Introduction

ORTHOGONAL frequency division multiplexing (OFDM) is a modulation scheme that is capable of overcoming intersymbol interference (ISI) on frequency-selective channels in a very efficient way. A single digital information stream is divided into multiple information streams. This lowers the bit rate on each of these streams. The datastreams are modulated and mapped on orthogonal carriers (called "subcarriers"). Thus, many low bit-rate signals are transmitted, instead of one high bit-rate signal. The low bit-rate signals hardly suffer from ISI and, because of the orthogonality of the subcarriers, it is possible to demodulate the received signal without crosstalk between the information on the subcarriers. The performance of OFDM can approach the theoretical maximum for a given radio channel [1].

A disadvantage of OFDM is its sensitivity to nonlinear distortion, which causes crosstalk between subcarriers. In a matched-filter receiver, the crosstalk can dramatically increase the symbol error rate (SER). As any real communication system will contain nonlinearities, it is important to determine the resulting signal deterioration. In particular, the antenna output amplifier of a transmitter can cause significant nonlinear distortion.

So far, the distortion problem for matched-filter receivers has been treated in two ways: either by simulation [2], [3] or by mathematical analysis [4]. The simulation results do not yield much insight into the problem. The reported analytical results in literature apply to specific clipping nonlinearities only and cannot be used for design purposes. The effect of smooth nonlinear distortion on matched-filter reception of an OFDM signal is more relevant, as it is the most common type of distortion.

Paper approved by C. Tellambura, the Editor for Modulation and Signal Design of the IEEE Communications Society. Manuscript received March 21, 2000; revised February 6, 2001. This work was supported by the Ubiquitous Communications Program, Delft University of Technology, The Netherlands. This paper was presented in part at the ISCAS 2000, Geneva, Switzerland, May 28–31, 2000, and at ICC2000, New Orleans, LA, June 18–22, 2000.

C. van den Bos and W. A. Serdijn are with the Electronics Research Laboratory, Delft University of Technology, 2628 CD Delft, The Netherlands (e-mail: c.vandenbos@its.tudelft.nl).

M. H. L. Kouwenhoven is with National Semiconductor, 2628 CD Delft, The Netherlands.

Publisher Item Identifier S 0090-6778(01)08160-0.

In this letter, analytical results are presented which apply to distortion of bandpass OFDM signals due to arbitrary memoryless, smooth nonlinear transfers. They enable a designer to estimate the influence of nonlinearity before performing simulations.

The organization of this letter is as follows. In Section II, the model for nonlinear transfer is discussed and the relations with common electronic performance measures such as IP2 and IP3 are given. In Section III, the signal-to-distortion ratio (SDR) due to nonlinearity is calculated. In Section IV, an example for QPSK-OFDM is treated both by analysis and simulation, to show the validity of the distortion model. In Section V, conclusions are presented.

II. Smooth Nonlinearity Model

Nonlinear distortion of OFDM signals causes the loss of orthogonality of subcarriers, and thereby inter-carrier interference. The result is an increase of the SER when the OFDM signal is detected by a matched-filter receiver.

The analysis in this letter is restricted to smooth nonlinear transfers. These are likely to be dominant and mainly originate in the OFDM transmitter output amplifier. A suitable first (design) approximation of such an amplifier is a memoryless nonlinearity.

The nonlinear distortion performance of amplifiers is often expressed in terms of intercept points (IP2, IP3), or intermodulation points (IM2, IM3). The parameters of the model have to be related to these measures. As intercept points and intermodulation points yield the same information for a memoryless nonlinearity, only the relation between intercept points and the model parameters is given.

The output signal $y(t) = f[s(t)]$ of a memoryless nonlinearity at a certain time instant only depends on the input signal at that same time instant. As $f(s)$ is smoothly nonlinear, it can be accurately represented by a few (K) terms of a Taylor polynomial as

$$f(s) \approx \sum_{k=0}^{K-1} a_k s^k \text{ where } a_k = \frac{1}{k!} \frac{\partial^k f(s)}{\partial s^k}\bigg|_{s=0}. \quad (1)$$

According to [5], intercept points and the model parameters are related by

$$\text{IP2} = \frac{2a_1}{a_2} \text{ and } \text{IP3} = \sqrt{\frac{4a_1}{a_3}}. \quad (2)$$

It is important to note that even-order distortion (k even) is mapped to bands that are far from the original OFDM signal passband. It is assumed that these are filtered out and do not influence the SER. Therefore, only odd-order distortion is of importance.

III. Signal Distortion Analysis

In this section, the effect of distortion on the SER of an OFDM signal is determined. First, the spectrum of a distorted OFDM signal is examined. It is used to determine the detection error for matched-filter detection. After this, the SER can be determined.

The complex lowpass OFDM signal is described by [6]

$$s_l(t) = \sum_{n=0}^{N-1} d_n(t) e^{j2\pi nt/T} \tag{3}$$

d_n complex-valued digital symbols that are constant during T seconds;
N number of complex subcarriers;
$e^{j2\pi nt/T}$ complex-valued orthogonal subcarriers.

N is typically larger than 100. The d_n carry information and are therefore random variables. They are assumed to be independent, of zero mean, and of identical distribution. The same modulation is used on each subcarrier. In practice, $s_l(t)$ is often formed by applying an IFFT to the symbols d_n.

The power spectrum of an OFDM bandpass signal is approximately rectangular and occupies a bandwidth of N/T Hz. The amplitude distribution of an OFDM signal is nearly Gaussian. This can be understood by noting that an OFDM signal consists of a sum of a large number of independent, identically distributed signals, and applying the Central Limit Theorem. The fact that the amplitude distribution is Gaussian is used for determining the spectrum of the distorted OFDM signal.

In the remainder of this letter, only third-order distortion is taken into account for SER calculation, because for smooth nonlinearities third-order distortion is often dominant. Even if it is not, inclusion of only third-order distortion will give a good estimate of the SER to be expected. If necessary, the procedure that is followed can also be used for calculating the influence of higher order distortion.

A. Distortion Spectral Analysis

In this section, the spectrum of the third-order distortion of the OFDM signal is calculated. The result is used to calculate the distortion variance after matched filtering. After matched-filter detection, the OFDM signal and the distortion are nearly uncorrelated, which can be explained as follows. Due to the distortion, orthogonality between subcarriers is lost. Each OFDM subcarrier symbol will contribute to other subcarrier symbols and itself. However, the number of contributions due to other subcarriers will outnumber the contribution of any subcarrier to itself, as there are many subcarriers. Thus, the correlation between OFDM signal and distortion, $R_{s^3s}(\tau)$, after matched-filter detection is nearly zero and can be neglected.

The spectrum is calculated through the autocorrelation function which is

$$R_{s^3}(\tau) = R_{s^3}(t_2 - t_1) = E\left(s_{t_1}^3 s_{t_2}^3\right). \tag{4}$$

The amplitude distribution of an OFDM signal is approximately Gaussian, and (4) can be evaluated employing a theorem from [7], p343. In case of third-order distortion, it is found that

$$R_{s^3}(\tau) = 6R_s^3(\tau) + 9\sigma_s^4 R_s(\tau). \tag{5}$$

A general expression for the autocorrelation function of s^n is

$$R_{s^n}(\tau) = \sum_{\substack{k=0\\(n-k)\text{even}}}^{n} \frac{(n!)^2}{k!\left[\left(\frac{n-k}{2}\right)!\right]^2 2^{n-k}} R_s^k(\tau)\sigma_s^{2(n-k)}. \tag{6}$$

The autocorrelation function and the power spectral density are related by

$$\begin{aligned} S_{s^3}(f) &= \int_{-\infty}^{\infty} R_{s^3}(\tau)e^{-j2\pi f\tau}d\tau \\ &= 6S_s(f) * S_s(f) * S_s(f) + 9\sigma_s^4 S_s(f). \end{aligned} \tag{7}$$

To evaluate the convolution, knowledge of the original OFDM spectrum is required. It is assumed that the OFDM spectrum is rectangular, with power σ_s^2 and nonzero for $f_0 - B < |f| < f_0 + B$. Although in reality there are sidelobes, the signal power in these is small enough to be neglected in the calculation of the effect of distortion on the SER, because the distortion components due to the rectangular part of the signal spectrum will be much stronger.

Using an expression for threefold convolution of a rectangular spectrum, adopted from [8], it is found that the bandpass spectrum of the distortion that is uncorrelated with the OFDM signal after detection is

$$\begin{aligned} S_{s^3}(f) = {} & \frac{9\sigma_s^6}{4B}a_3^2 \begin{cases} 1, & |f \pm f_0| < B \\ 0, & |f \pm f_0| > B \end{cases} + \frac{3\sigma_s^6}{32B^3}a_3^2 \\ & \cdot \begin{cases} 3B^2 - (f \pm f_0)^2, & |f \pm f_0| < B \\ \frac{1}{2}(3B - |f \pm f_0|)^2, & B < |f \pm f_0| < 3B \\ 0, & |f \pm f_0| > 3B \end{cases} \end{aligned} \tag{8}$$

which is used to calculate the SDR.

B. Influence of Distortion on Correlation Reception

In this section, the influence of the distortion on the SER is investigated. In many practical receivers, matched-filter detectors are used. In case of Gaussian noise, a matched-filter detector is optimal.

However, the distribution of the distortion is not Gaussian, and therefore the distribution of the distortion after matched filtering cannot be assumed to be Gaussian. The exact calculation of the distribution is a difficult problem.

Fortunately, a result by Papoulis [9] states that a random process that is narrow-band filtered has an output amplitude distribution that approaches a Gaussian distribution. As the OFDM matched filters are narrow-band with respect to the signal and the distortion, it can be concluded that the distortion must be approximately Gaussian. Further, it can be shown that the distortion on $\mathrm{Re}(d_n)$ and $\mathrm{Im}(d_n)$ of a subcarrier is uncorrelated.

Knowing the distortion at the output of the matched filter is Gaussian, it suffices to calculate the mean and the variance of the distortion on each subcarrier to completely characterize the stochastic process. As only the third-order distortion contributes to in-band distortion components, an amplifier model with only third-order distortion is used, so only $a_1, a_3 \neq 0$ in (1).

The relation between input and output spectrum of a filter, matched to OFDM carrier n, is given by

$$S_y(f) = S_s(f)\mathrm{sinc}^2\{\pi[n - (f - f_0)]\} \tag{9}$$

where $\mathrm{sinc}(x) \equiv \sin(x)/x$. The distortion spectrum is given by (8). To determine the distortion power, the variance of $S_y(f)$

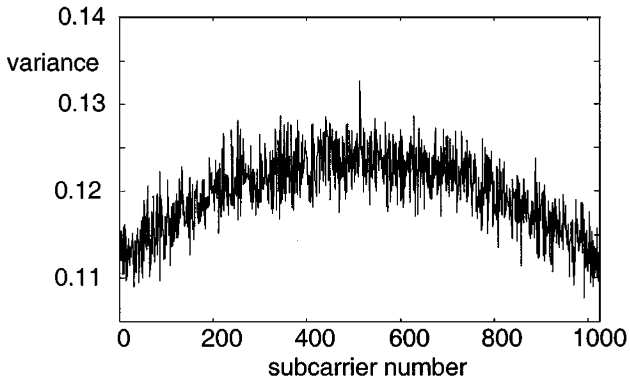

Fig. 1. The variance per subcarrier from simulation: $N = 1024$, $a_3 = -0.1$, $\sigma_x^4 = 1$, average for 1000 OFDM symbols.

in (9) has been calculated for each subcarrier. Two calculations have been performed: one including the distortion outside the principal OFDM bandwidth N/T, and one excluding this distortion. There was no significant difference between the two results.

It found that the variance is nearly the same for each subcarrier, so the distortion is (nearly) evenly spread over all subcarriers (see also Fig. 1). The bit error rates on all subcarriers are therefore nearly equal.

The input distortion variance is found by integrating the distortion power spectrum, (8), over the OFDM bandwidth. Straightforward calculation yields an in-band distortion power of $10a_3^2\sigma_s^6$.

The signal power after the nonlinearity is $a_1^2\sigma_s^2$, so the SDR is given by

$$\mathrm{SDR} = \left(\frac{a_1}{a_3}\right)^2 \frac{1}{10\sigma_s^4} \tag{10}$$

where SDR is the signal-to-uncorrelated-distortion ratio. The same SDR is found at the output of a matched filter, because the spectra of the OFDM signal and uncorrelated distortion have the same shape in the frequency region of interest.

IV. Example

In this section, the effect of distortion on QPSK-OFDM and QAM16-OFDM is examined by means of analytical calculation and by simulation. The bandpass signals pass through a nonlinear amplifier and are detected by a matched-filter detector. Subsequently, the distortion variance is determined by means of simulation and calculation, and a comparison is made. Finally the SER is determined.

If the OFDM signal power is given by σ_s^2 and the amplifier transfer is $y = s + a_3 s^3$, the SDR is given by

$$\mathrm{SDR} = \frac{1}{10a_3^2\sigma_s^4}. \tag{11}$$

The SDR at the output of the matched filter is the same as the SDR at the input. The average energy of a QPSK or QAM16 symbol E_s is set to 1, so the average power per subcarrier is also 1. Thus, the distortion power per subcarrier should be $\sigma_d^2 = P_{\mathrm{QPSK}}/\mathrm{SDR} = 1/\mathrm{SDR}$. For $\sigma_s^2 = 1$, the calculation has been performed for several values of a_3. The results are summarized in Table I for easy comparison to simulation results. IP3 has also been included.

TABLE I
CALCULATED AND SIMULATED DISTORTION POWER PER SUBCARRIER; $a_1 = 1$

a_3	IP3(dBm)	distortion power (calculated)	distortion power (simulated)
-0.1	16	0.10	0.119
-0.01	26	$1.0 \cdot 10^{-3}$	$1.21 \cdot 10^{-3}$

The simulations have been performed using MATLAB. To circumvent up-and downconversion to an RF frequency, an equivalent low-pass representation of the nonlinearity has been used, which is allowed for in-band distortion.

An OFDM system with $N = 1024$ carriers is considered. The OFDM modulator performs an IFFT on a complex input vector. The input vector elements represent the QPSK or QAM16 symbols. The output vector of the IFFT is processed by the nonlinearity. The output vector is demodulated by an FFT.

The symbols in the original and the demodulated vector are compared. The difference is squared and stored for each symbol separately. The procedure is repeated several times, and the results are averaged. Thus, for each subcarrier, the distortion variance can be depicted. An example is shown in Fig. 1.

The symbol error rate (SER) vs a_3 curve for QPSK is shown in Fig. 2. It can be seen that there is a sharp drop in the simulated SER for $a_3 \approx -0.11$. The theoretical symbol error rate is given by [10]

$$P_{\mathrm{SER,QPSK}} = \mathrm{erfc}\left(\sqrt{\frac{\mathrm{SDR}}{2}}\right). \tag{12}$$

For QAM16, the SER is shown in Fig. 3. The theoretical symbol error rate is given by

$$P_{\mathrm{SER,QAM16}} = \frac{3}{2}\mathrm{erfc}\left(\sqrt{\frac{\mathrm{SDR}}{10}}\right). \tag{13}$$

It is seen that the simulation results for QAM16 are closer to the theoretical curve than are those for QPSK. This is because QAM16 has more symbols at different symbol energy levels, i.e., the QAM16-OFDM distribution approaches Gaussianity more quickly than the QPSK-OFDM distribution. It can also be noted that in both cases, the simulated SER curves drop below the theoretical curve. This is because the distortion distribution is not exactly Gaussian, and the tail distribution falls off more quickly than in the real Gaussian case. Still, the simple expression for the distortion can well be used to estimate the minimum allowable IP3.

Previously, only the effect of distortion and Gaussian noise together has been studied [2], [4]. When noise and distortion are present, they both contribute to errors, thereby masking the influence of the distortion alone. Therefore, the influence of distortion alone has been investigated in this letter. If necessary, the SER calculations in this letter can be extended by using the

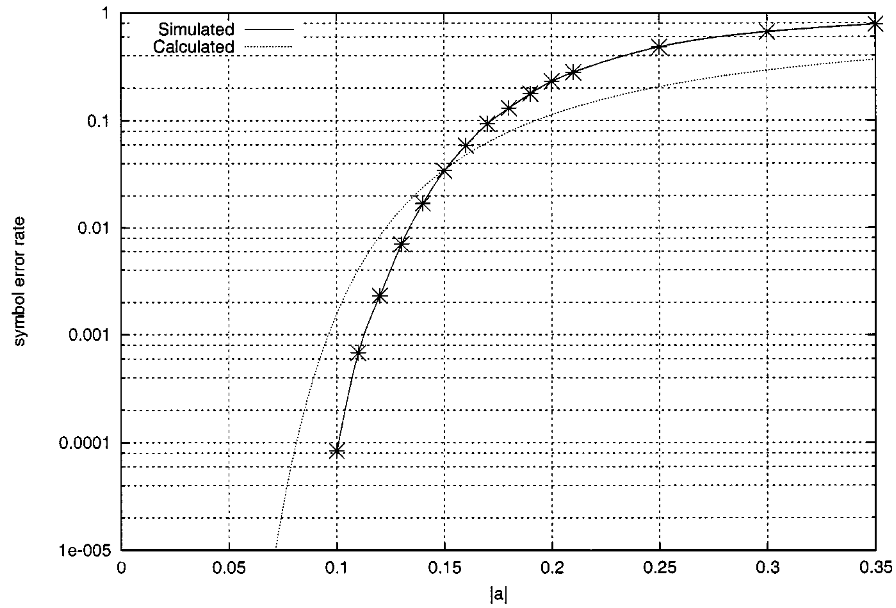

Fig. 2. The SER of a QPSK-OFDM signal as a function of third-order distortion. $N = 1024$.

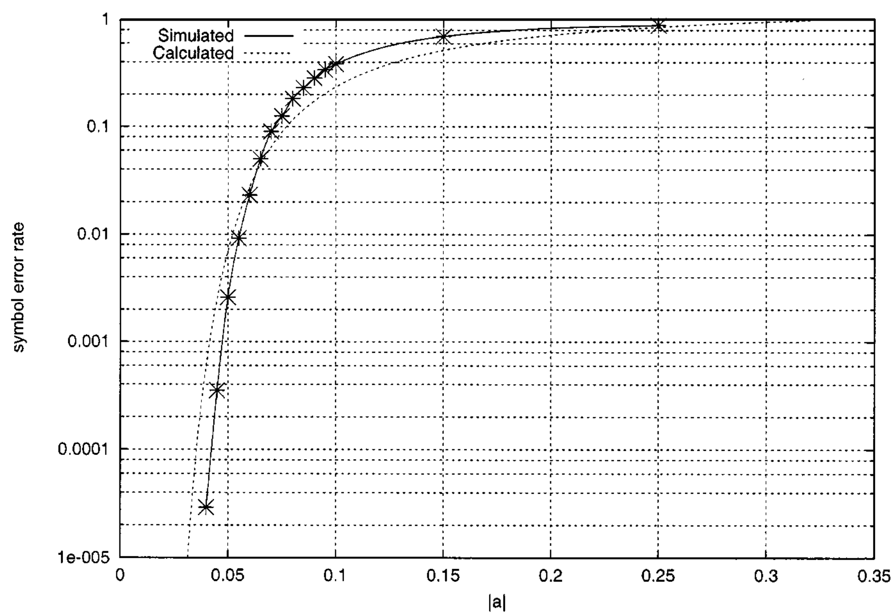

Fig. 3. The SER of a QAM16-OFDM signal as a function of third-order distortion. $N = 1024$.

signal-to-noise-and-distortion ratio (SNDR) instead of using the SDR [4]. The SNDR is given by

$$\frac{1}{\mathrm{SNDR}} = \frac{1}{\mathrm{SNR}} + \frac{1}{\mathrm{SDR}} \tag{14}$$

where SNR is the signal-to-noise ratio without distortion.

V. Conclusion

In this letter, the effect of memoryless, smooth nonlinear distortion on bandpass OFDM signals has been investigated. In particular, the OFDM power spectrum and the bit error rate after linear detection have been analyzed. The method employed is applicable to any memoryless, smooth nonlinear distortion, so it is not restricted to special cases only.

Only the effect on third-order distortion has been taken into account. Inclusion of third-order distortion is sufficient for a good first approximation. If needed, the analysis can readily be extended to include higher-order distortion as well.

The results are simple analytical expressions, which can be evaluated by manual calculation. This is particularly interesting to designers who would like to predict system behavior before performing any simulations. Comparison of analytical and simulation results shows that the simple expressions can be used to estimate the minimum allowable IP3.

The signal deterioration has been examined for each separate subcarrier, which revealed that the error on each subcarrier is approximately equal. This means that as far as distortion is concerned, it does not make sense to distribute data over subcarriers in a particular way to increase the overall performance.

REFERENCES

[1] J. A. C. Bingham, “Multicarrier modulation: An idea whose time has come,” *IEEE Commun. Mag.*, vol. 28, pp. 5–14, May 1990.

[2] S. Merchán, A. G. Armada, and J. García, “OFDM performance in amplifier nonlinearity,” *IEEE Trans. Broadcast.*, vol. 44, pp. 106–114, Mar. 1998.

[3] X. Li and L. J. Cimini, “Effects of clipping and distortion on the performance of OFDM,” *IEEE Commun. Lett.*, vol. 2, pp. 131–133, May 1998.

[4] Q. Shi, “OFDM in bandpass nonlinearity,” *IEEE Trans. Consumer Electron.*, vol. 42, pp. 253–258, Aug. 1996.

[5] B. Gilbert, “Design considerations for bjt active mixers,” in *Low-Power HF Microelectronics: A Unified Approach*. ser. IEE Circuits and Systems Series 8, G. A. S. Machado, Ed. London, U.K.: Inst. Elect. Eng., 1996.

[6] O. Edfors, M. Sandell, J. van de Beek, D. Landstrom, and F. Sjoberg. (1996) *An Introduction to Orthogonal Frequency-Division Multiplexing*. [Online]. Available: http://www.sm.luth.se/csee/sp/publications/

[7] D. Middleton, *An Introduction to Statistical Communication Theory*. New York: McGraw-Hill, 1960.

[8] B. F. McGuffin, “The effect of bandpass limiting on a signal in wideband noise,” in *Proc. MILCOM*, New York, 1992, pp. 206–210.

[9] A. Papoulis, “Narrow-band systems and gaussianity,” *IEEE Trans. Inform. Theory*, vol. IT-18, pp. 20–27, Jan. 1972.

[10] J. G. Proakis, *Digital Communications*, 3rd ed. Singapore: McGraw-Hill, 1995.