The influence of non-linear distortion on OFDM bit error rate

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Abstract—**The influence of bandpass non-linear distortion on an OFDM signal is investigated. The spectrum of the distorted signal, and the resulting detection errors are determined. Non-linear distortion of OFDM signals can significantly increase the receiver BER. Previous analyses rely on simulation, or apply to specific non-linearities only. Here, an analysis is presented which is applicable to any weakly non-linear distortion of an OFDM signal. Simple analytical expressions are derived which allow a designer to choose parameters without performing simulations. Further, it is shown that the error on each OFDM subcarrier is approximately equally large.**

I. Introduction

OFDM (Orthogonal Frequency Division Multiplexing) is a modulation scheme that is capable of overcoming intersymbol interference (ISI) on frequency selective channels in a very efficient way. A digital information stream is divided into multiple information streams. This lowers the bitrate on each of these streams. The datastreams are modulated and mapped on orthogonal carriers (called "subcarriers"). Thus, many low-bitrate signals are transmitted, instead of one high bitrate signal. The low bitrate signals hardly suffer from ISI, and because of the orthogonality of the subcarriers, it is possible to demodulate the received signal without crosstalk between the information on the subcarriers. The performance of OFDM can approach the theoretical maximum for a given radio channel, [1].

A disadvantage of OFDM is its sensitivity to non-linear distortion. Non-linear distortion causes loss of orthogonality between subcarriers, resulting in crosstalk. In a matched-filter receiver, the crosstalk can dramatically increase the BER. As any real communication system will contain non-linearities, it is important to determine the resulting signal deterioration. Especially the antenna output amplifier of a transmitter can cause significant non-linear distortion.

So far, the distortion problem for matched filter receivers has been treated in two ways: either by simulation, [2], [3], or by mathematical analysis, [4]. The simulation results do not yield much insight into the problem. The reported analytical results in literature apply to specific clipping non-linearities only and cannot be used for design purposes. The effect of smooth non-linear distortion on matched filter reception of an OFDM signal is more relevant, as it is the most common type of distortion.

In this paper, analytical results are presented which apply to distortion of bandpass OFDM signals due to arbitrary memoryless, smooth non-linear transfers. They enable a designer to estimate the influence of non-linearity before performing simulations.

The organization of this paper is as follows. In section II, the model for non-linear transfer is discussed and the relations with common electronic performance measures such as IP2 and IP3

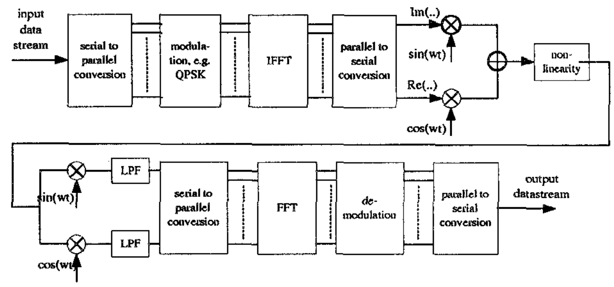

Fig. 1. Practical OFDM system implementation

are given. In section III, the SDR (signal-to-distortion ratio) due to non-linearity is calculated. Subsequently the BER after detection by a matched filter receiver is calculated. In section IV, an example for QPSK-OFDM is treated both by analysis and simulation, to show the validity of the distortion model. In section V, conclusions are presented.

II. Models for smooth non-linearities

Non-linear distortion of OFDM signals causes the loss of orthogonality of subcarriers, and thereby inter-carrier interference. The result is an increase of the symbol-error rate when the OFDM signal is detected by a matched-filter receiver.

In figure 1, a practical OFDM system including non-linearity is shown. In this system, an IFFT and FFT are used for modulation and demodulation. Essentially, it is a discrete-time system. As the bandpass OFDM signal is continuous, a continuous-time model is used throughout this paper. In figure 2, the corresponding system is depicted.

The analysis in this paper is restricted to smooth non-linear transfers. These are likely to be dominant, and mainly originate in the OFDM transmitter output amplifier. A suitable approximation of such an amplifier is a memoryless non-linearity.

To quantify the effect of non-linear distortion a suitable mathematical model must be used. The non-linear distortion performance of amplifiers is often expressed in terms of intercept points (IP2, IP3), or intermodulation points (IM2, IM3). The parameters of the model have to be related to these measures. It can be shown that intercept points and intermodulation points yield the same information for a memoryless non-linearity. Therefore only the relation between intercept points and the model parameters is given.

The output signal of a memoryless non-linear system at a certain time instant only depends on the input signal at that same time instant. The input-output relation of such a system is then

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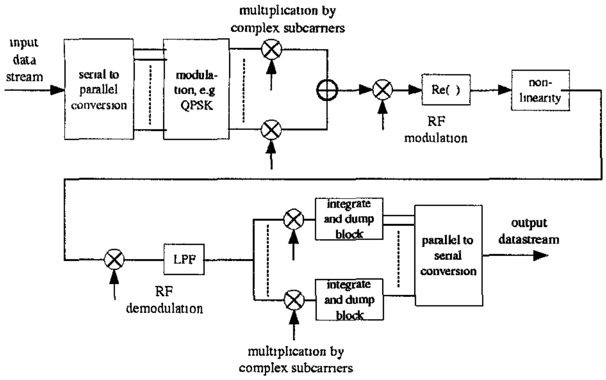

Fig. 2. Equivalent OFDM system, which is employed for analysis

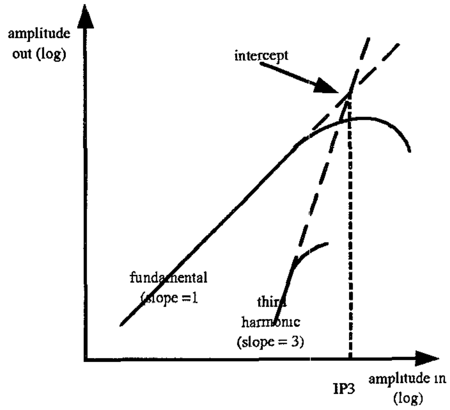

Fig. 3. Determination of the intercept point IP3

given by

$$y(t) = f[x(t)] \tag{1}$$

i.e. depending on time implicitly. As $f(x)$ is smoothly non-linear, it can be accurately represented by a few (K) terms of a Taylor polynomial:

$$f(x) \approx \sum_{k=0}^{K-1} a_k x^k, \qquad a_k = \frac{1}{k!}\frac{\partial^k f(x)}{\partial x^k}\bigg|_{x=0} \tag{2}$$

It is important to note that even-order distortion (k even) is mapped to bands that are far from the original OFDM signal passband. It is assumed that these are filtered out and are do not influence the BER. Therefore, only odd-order distortion is of importance.

Intercept points IPk are distortion measures for a single sinusoidal signal at the input of the non-linearity. The intercept points are found by extrapolating fundamental and harmonic input-output amplitude curves from the signal range where little distortion occurs. The kth order input referred intercept point is defined as the input signal amplitude at which the amplitude of the linear signal component, and the kth order harmonic distortion component would become equal [5]. In reality, this situation does not occur, because the linear transfer and the distortion curves are not straight lines (figure 3).

IP2 and IP3 are related to the model of the non-linearity as follows. Taking $a_k = 0$ for $k > 3$, only second and third order distortion are included. At the input of the non-linearity, there is a signal $x(t) = A\sin(\omega t)$. According to equation (2), this gives an output signal

$$\begin{aligned} y(t) &= \sum_{k=0}^{K} a_k x^k(t) = \sum_{k=0}^{3} a_k A^k \sin^k(\omega t) = \\ &\left(a_0 + \frac{1}{2}a_2A^2\right) + A\left(a_1 + \frac{3}{4}a_3A^2\right)\sin(\omega t) \\ &\quad - \frac{1}{2}a_2A^2\cos(2\omega t) - \frac{1}{4}a_3A^3\sin(3\omega t) \end{aligned} \tag{3}$$

Thus the second-order intercept point IP2 is given by

$$A = \text{IP2} = \frac{2a_1}{a_2} \tag{4}$$

and the third-order intercept point IP3 is given by

$$A = \text{IP3} = \sqrt{\frac{4a_1}{a_3}} \tag{5}$$

The relation between the third-order intercept point and the model parameters will prove to be of use for the determination of the influence of distortion on linear detector performance.

III. Signal distortion analysis

In this section, the effect of distortion on the BER of an OFDM signal is determined. First, the spectrum of a distorted OFDM signal is examined. It is used to determine the detection error for matched filter detection. It is shown that the error has a (approximate) gaussian amplitude distribution, and is approximately uncorrelated with the OFDM signal itself. After this, the BER can be determined.

The complex bandpass OFDM signal is described by [6]

$$s_l(t) = \sum_{n=0}^{N-1} d_n(t) e^{j2\pi nt/T} \tag{6}$$

d_n are complex-valued digital symbols that are constant during T seconds, N represents the number of complex subcarriers, and $e^{j2\pi nt/T}$ represent complex-valued orthogonal subcarriers. N is typically larger than 100. The d_n carry information and are therefore random variables. They are assumed to be independent, of zero mean, and of identical distribution, so the same modulation (e.g. QPSK) on each subcarrier. In practice, $s_l(t)$ is often formed by performing an FFT on the symbols d_n.

The power spectrum of an OFDM bandpass signal is approximately rectangular, and occupies a bandwidth of N/T Hz. The amplitude distribution of an OFDM signal is nearly gaussian.

This can be understood by noting that an OFDM signal consists of a sum of a large number of independent, identically distributed signals, and applying the Central Limit Theorem. The fact that the amplitude distribution is gaussian is used for determining the spectrum of the distorted OFDM signal, [7].

In the remainder of this paper, only third-order distortion is taken into account for BER calculation, because for smooth nonlinearities third-order distortion is often dominant. Even if it is not, inclusion of third-order distortion only will give a good estimate of the BER to be expected.

A. Distortion spectral analysis

In this section, the spectrum of the third power of the OFDM signal only is calculated, equation (2). The result is used to calculate the distortion variance after matched filtering. After matched-filter detection, the OFDM signal and the distortion are nearly uncorrelated. Without going into lengthy mathematical details, this can be explained as follows. Due to the distortion, orthogonality between subcarriers is lost, so each OFDM subcarrier symbol will contribute to other subcarrier symbols and itself. The number of contributions due to other subcarriers will outnumber the contribution of any subcarrier to itself. Therefore, the correlation between OFDM signal and distortion is not calculated.

The spectrum is calculated through the autocorrelation function which is

$$R_{x^3}(\tau) = R_{x^3}(t_2 - t_1) = E(x_{t_1}^3 x_{t_2}^2) \tag{7}$$

This expression is evaluated using a theorem for stochastic processes that are gaussian in distribution, [7].

Theorem 1: If $n_1, \dots, n_{2k}$ denote an even number of zero-mean Gaussian random variables, not necessarily independent, then

$$E(n_1 \dots n_{2k}) = \sum_{\text{all pairs } i \neq j} \prod^{k} E(n_i n_j) \tag{8}$$

As an example, suppose a gaussian signal $x(t)$ is squared, so $y(t) = x^2(t)$. The autocorrelation function of y is given by

$$R(t_1, t_2) = E(y_{t_1} y_{y_2}) = E(x_{t_1} x_{t_1} x_{t_2} x_{t_2}) \tag{9}$$

From the theorem, for $k = 2$, it follows

$$\begin{aligned} E(n_1 n_2 n_3 n_4) &= E(n_1 n_2) E(n_3 n_4) && (10)\\ &+ E(n_1 n_3) E(n_2 n_4) && (11)\\ &+ E(n_1 n_4) E(n_2 n_3) && (12) \end{aligned}$$

Substituting $n_1 = x_{t_1}$ and $n_2 = n_3 = n_4 = x_{t_2}$ it is found that

$$E(x_{t_1} x_{t_1} x_{t_2} x_{t_2}) = E(x_{t_1}^2) E(x_{t_2}^2) + 2E^2(x_{t_1} x_{t_2}) \tag{13}$$

so the desired correlation function is

$$R_y(\tau) = \sigma_x^4 + 2R_x^2(\tau) \tag{14}$$

where $\tau = t_1 - t_2$. A similar calculation can be performed for $E(x_{t_1}^3 x_{t_2}^3)$, yielding

$$R_{x^3}(\tau) = 6R_x^3(\tau) + 9\sigma_x^4 R_x(\tau) \tag{15}$$

which is the desired autocorrelation function.

The autocorrelation function and the power spectral density are related by

$$S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-j2\pi f \tau} d\tau \tag{16}$$

Transformation of terms involving $R_x(\tau)$ is trivial. Transformation of $R_x^3(\tau)$ gives a threefold convolution:

$$S(f) = \int_{-\infty}^{\infty} R_x^3(\tau) e^{-j2\pi f \tau} d\tau = S_x(f) * S_x(f) * S_x(f) \tag{17}$$

To evaluate this convolution, knowledge of the original OFDM spectrum is required. Here it is assumed that the OFDM spectrum is rectangular, which is a good approximation of the real situation. For a rectangular spectrum described by

$$F_1(f) = \frac{\sigma_x^2}{2B} \begin{cases} 1 & |f| \leq B \\ 0 & |f| > B \end{cases} \tag{18}$$

the following expressions give the twofold and the threefold convolution of the spectrum with itself, [8].

$$F_2(f) = \frac{\sigma_x^4}{4B^2} \begin{cases} 2B - |f| & |f| < 2B \\ 0 & |f| \geq 2B \end{cases} \tag{19}$$

$$F_3(f) = \frac{\sigma_x^6}{8B^3} \begin{cases} 3B^2 - f^2 & |f| < B \\ \frac{1}{2}(3B - |f|)^2 & B < |f| \leq 3B \\ 0 & |f| > 3B \end{cases} \tag{20}$$

The bandpass OFDM signal spectrum is

$$S_{\text{bandpass}}(f) = \frac{1}{2} F_1(f - f_0) + \frac{1}{2} F_1(f + f_0) = \frac{1}{2} F_1(f) * [\delta(f - f_0) + \delta(f + f_0)] \tag{21}$$

so for $R_x^3(\tau)$,

$$S_y(f) = \frac{1}{8} F_3(f) * [\delta(f - f_0) + \delta(f + f_0)]_{\text{threefold convolved}} \tag{22}$$

so the spectrum of the distortion that is uncorrelated with the OFDM signal after detection is

$$S_y(f) = \frac{9\sigma_x^6}{4B} a_1^2 \begin{cases} 1 & |f \pm f_0| < B \\ 0 & |f \pm f_0| > B \end{cases} + \frac{3\sigma_x^6}{32B^3} a_3^2 \begin{cases} 3B^2 - (f \pm f_0)^2 & |f \pm f_0| < B \\ \frac{1}{2}(3B - |f \pm f_0|)^2 & B < |f \pm f_0| < 3B \\ 0 & |f \pm f_0| > 3B \end{cases} \tag{23}$$

which is used to calculate the SDR.

B. *Influence of distortion on correlation reception*

In this section, the influence of the distortion on the BER is investigated. In many practical receivers, matched filter detectors are used. In case of Gaussian noise, a matched filter detector is optimal.

However, the distribution of the distortion is not gaussian, and therefore the distribution of the distortion after matched filtering cannot be assumed to be gaussian. The exact calculation of the distribution is a difficult problem, hampering the calculation of the BER.

Fortunately, a result by Papoulis, [9] states that a random process that is narrowband filtered has an output amplitude distribution that approaches a gaussian distribution. As the OFDM matched filters are narrowband with respect to the signal and the distortion, it can be concluded that the distortion must be approximately gaussian. Further, it can be shown that the distortion on $\mathrm{Re}(d_n)$ and $\mathrm{Im}(d_n)$ of a subcarrier is uncorrelated.

Knowing the distortion at the output of the matched filter is gaussian, it suffices to calculate the mean and the variance of the distortion on each subcarrier to completely characterize the stochastic process. As only the third-order distortion contributes to in-band distortion components, an amplifier model with only third-order distortion is used:

$$y = a_1 x + a_3 x^3 \tag{24}$$

The relation between input and output spectrum of a filter is given by

$$S_y(f) = S_x(f)|H(f)|^2 \tag{25}$$

where $H(f)$ is the transfer function of the filter.

When $H(f)$ is a matched filter for OFDM, it is a sinc-function with the peak at the frequency of the desired subcarrier, so for subcarrier n:

$$|H(f)|^2 = \{\mathrm{sinc}[\pi(n-(f-f_0)T)]\}^2 \tag{26}$$

where $\mathrm{sinc}(x) \equiv \frac{\sin(x)}{x}$. The distortion spectrum is given by equation (23). For a simple example, $T = 1$ and $N = 200$. To determine the distortion power, the variance of $S_x(f)|H(f)|^2$ is calculated for each subcarrier. The result is depicted in figure 4. Two calculations have been performed: one including the distortion outside the principal OFDM bandwidth N/T, and one excluding this distortion. There was no significant difference between the two graphs. Therefore it is not useful to filter out the distortion outside the principal OFDM band.

It is seen that the variance is nearly the same for each subcarrier, so the distortion is (nearly) evenly spread over all subcarriers. The bit error rates on all subcarriers are therefore nearly equal.

The input distortion variance is found by integrating the distortion power spectrum. Straightforward calculation yields an in-band distortion power of

$$10a_3^2\sigma_x^6 \tag{27}$$

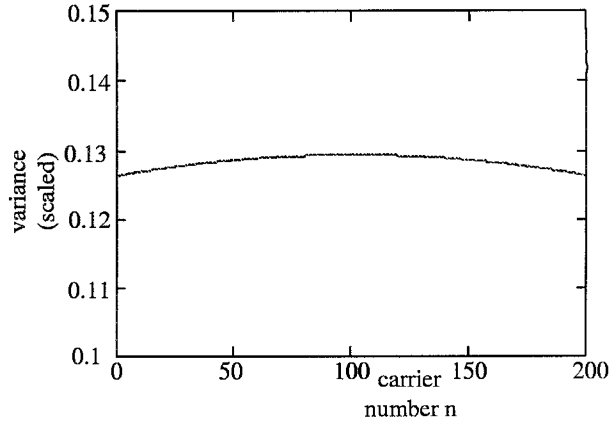

Fig. 4. Shape of variance of distortion on OFDM subcarriers, $N = 200$.

The signal power is $a_1^2\sigma_x^2$, so the SDR is given by

$$\mathrm{SDR} = \left(\frac{a_1}{a_3}\right)^2 \frac{1}{10\sigma_x^4} \tag{28}$$

where SDR is the signal to uncorrelated distortion ratio. The same SDR is found at the output of a matched filter, because the spectra of OFDM signal and uncorrelated distortion have the same shape in the frequency region of interest.

IV. Example

In this section, an example is discussed. A bandpass QPSK-OFDM signal passes through a non-linear amplifier and is detected by a matched filter detector. Subsequently, the distortion variance is determined by means of simulation and calculation, and a comparison is made. Finally the BER is determined.

If the OFDM signal power is given by σ_x^2 and the amplifier transfer is $y = x + a_3x^3$, the SDR is given by

$$\mathrm{SDR} = \frac{1}{10a_3^2\sigma_x^4} \tag{29}$$

The SDR at the output of the matched filter is the same as the SDR at the input. The energy of a QPSK symbol E_s is set to 1, so the QPSK power per subcarrier is also 1. Thus, the distortion power per subcarrier should be $\sigma_d^2 = \frac{P_{\mathrm{QPSK}}}{SDR} = \frac{1}{SDR}$. Thus, we have that

$$\sigma_n{}^2 = 10a_3^2\sigma_x^4 \tag{30}$$

For $\sigma_x^2 = 1$, the calculation has been performed for several values of a. The results are summarized in table I for easy comparison to simulation results. IP3 has also been included; as no units have been used, IP3 is denoted in dB.

The simulations have been performed using MATLAB. Essentially, the system depicted in figure 1 has been implemented. To circumvent up-and downconversion to an RF frequency, an equivalent low-pass representation of the non-linearity has been used.

a_3	IP3(dBm)	distortion power (calculated)	distortion power (simulated)
-1	6	10	11.8
-0.1	16	0.10	0.119
-0.01	26	$1.0 \cdot 10^{-3}$	$1.21 \cdot 10^{-3}$

TABLE I

CALCULATED AND SIMULATED DISTORTION POWER PER SUBCARRIER; $a_1 = 1$.

The OFDM modulator performs an IFFT on a complex input vector. The input vector elements represent the QPSK symbols. The output vector of the IFFT is processed by the non-linearity. The output vector is demodulated by an FFT.

The QPSK symbols in the original and the demodulated vector are compared. The difference is squared and stored for each symbol separately. The procedure is repeated several times, and the results are averaged. Thus, for each subcarrier, the distortion variance can be depicted. An example is shown in figure 5.

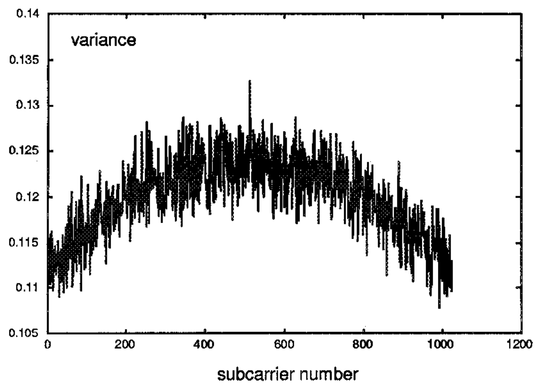

Fig. 5. The variance per subcarrier from simulation: $N = 1024$, $a = -0.1$, $\sigma_x^4 = 1$, average for 1000 OFDM symbols.

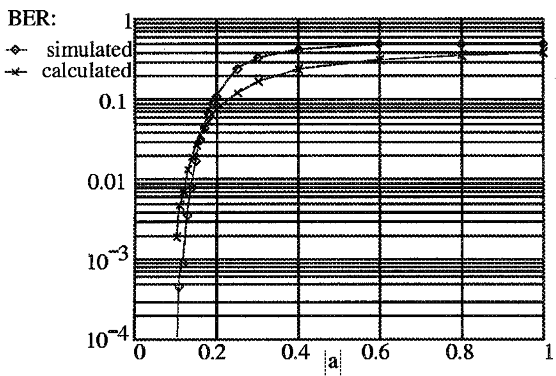

Fig. 6. The BER of a QPSK-OFDM signal as a function of third-order distortion

The BER vs a curve is shown in figure 6. It can be seen that there is a sharp drop in the simulated BER for $a_3 \approx -0.11$. This effect was not revealed previously, because the effect of distortion and gaussian noise together has been studied [2], [3]. From the graph it can be seen that the simple expression for the distortion can well be used to estimate the minimum allowable IP3.

V. CONCLUSIONS

In this paper, the effect of memoryless, smooth non-linear distortion on bandpass OFDM signals has been investigated. In particular, the OFDM power spectrum and the bit error rate after linear detection have been analyzed. The method employed is applicable to any memoryless, smooth non-linear distortion, so it is not restricted to special cases only.

Only the effect on third-order distortion has been taken into account. Inclusion of third-order distortion is sufficient for a good first approximation. If needed, the analysis can readily be extended to include higher-order distortion as well.

The results are simple analytical expressions, which can be evaluated by manual calculation. This is particularly interesting to designers who would like to predict system behaviour before performing any simulations. Comparison of analytical and simulation results shows that the simple expressions can be used to estimate the minimum allowable IP3.

The signal deterioration has been examined for each separate subcarrier, which revealed that the error on each subcarrier is approximately equal. This means that as far as distortion is concerned, it does not make sense to distribute data over subcarriers in a particular way to increase the overall performance.

VI. ACKNOWLEDGMENT

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