Equivalent Thévenin and Norton Kirchhoff circuits of a receiving antenna

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***Abstract*—The full equivalence of the Thévenin and Norton Kirchhoff circuits of an N-port receiving antenna is discussed.**

***Index Terms*—Thévenin, Norton, equivalent circuits, receiving antennas.**

I. Introduction

ONCE in a while, the equivalent Thévenin and Norton Kirchhoff circuits describing the interaction between a (passive) antenna and an electromagnetic (EM) field incident upon it from the embedding medium lead to discussions in the Antennas and Propagation literature [1]–[7]. In particular, the discussions focus on the power that is absorbed in the terminating loads in the relevant circuits. From the perspective of the wave motion in the embedding medium, the antenna system in its entirety acts as a *scatterer* composed of linear, time-invariant, locally reacting structure, for which the scattering problem has a unique solution [8]. As a consequence, the Thévenin and the Norton equivalent circuits should be each other's complete equivalence (notwithstanding apparent discrepancies) [3]. The latter property is the subject of discussion in the present note. Basically, the line of thought runs parallel to the one in the frequency-domain analysis [9] and to the one in the time-domain analysis [10].

The interfacing between the (global) field description in the embedding and the (local) Kirchhoff circuits is established by the Lorentz reciprocity relation of the time-convolution type [9], [11]. Let $\boldsymbol{E}, \boldsymbol{H}$ be the electric and the magnetic field in the embedding and let $[V]$ and $[I]$ denote the electric currents and the voltages in the Kirchhoff circuits as they are observed at the antenna's $N(N \geq 1)$ *ports of access*. For this theorem to hold, the antenna configuration should be linear, time-invariant and passive in its EM behavior, with locally reacting media.

The EM field quantities in the embedding and the Kirchhoff circuit quantities are expressed via their time Laplace transforms, with transform parameter $s \in \mathbb{C}$. Causality is ensured by taking $\Re(s) > 0$. From the expressions, the corresponding time-domain (TD) quantities follow by applying the standard rules that the product of two transformed quantities corresponds to time convolution, while the factor s corresponds to the operation of time differentiation. The corresponding frequency-domain (FD) expressions follow by taking $s = j\omega$, with j as the imaginary unit and $\omega \in \mathbb{R}$ as the angular frequency. The operation of time reversal is denoted by *. In the Laplace-transform domain, this operation corresponds to replacing s with $-s$ (which in the frequency domain amounts to taking the complex conjugate). The operation of the antenna in its *transmitting state* is denoted by ^{T}; the operation of the antenna in its *receiving state* is denoted by ^{R}.The incident wave in the embedding is denoted by ^{i} and the (outgoing) scattered wave by ^{s}. The spatial support of the antenna is the bounded domain $\mathcal{D} \subset \mathbb{R}^3$. The (unbounded) embedding then has the spatial support $\mathcal{D}^\infty$ where $\mathcal{D}^\infty$ is the complement of $\mathcal{D} \cup \partial\mathcal{D}$ in $\mathbb{R}^3$, where $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$, on which also the Kirchhoff circuit ports of access are located.

The interfacing expression from the Lorentz reciprocity relation is [9], [11]:

$$\sum_{n=1}^{N} \left(\hat{V}_n^T \hat{I}_n^R - \hat{V}_n^R \hat{I}_n^T \right) = \int_{\partial\mathcal{D}} \left(\hat{\boldsymbol{E}}^T \times \hat{\boldsymbol{H}}^R - \hat{\boldsymbol{E}}^R \times \hat{\boldsymbol{H}}^T \right) \cdot \boldsymbol{\nu} dA \tag{1}$$

or

$$\sum_{n=1}^{N} \left(\hat{V}_n^T \hat{I}_n^R - \hat{V}_n^R \hat{I}_n^T \right) = \int_{\partial\mathcal{D}} \left(\hat{\boldsymbol{E}}^T \times \hat{\boldsymbol{H}}^i - \hat{\boldsymbol{E}}^i \times \hat{\boldsymbol{H}}^T \right) \cdot \boldsymbol{\nu} dA, \tag{2}$$

where $\boldsymbol{\nu}$ is the unit vector along the outward normal to $\partial\mathcal{D}$, and the relation

$$\int_{\partial\mathcal{D}} \left(\hat{\boldsymbol{E}}^T \times \hat{\boldsymbol{H}}^s - \hat{\boldsymbol{E}}^s \times \hat{\boldsymbol{H}}^T \right) \cdot \boldsymbol{\nu} dA = 0 \tag{3}$$

has been used. (Note that both the transmitted field in the transmitting state and the scattered field in the receiving state are sourcefree in $\mathcal{D}^\infty$ and are outgoing at infinity. In accordance with the orientation of $\boldsymbol{\nu}$, the electric currents $\left[\hat{I}^T\right]$ and $\left[\hat{I}^R\right]$ are oriented into $\mathcal{D}^\infty$.

The relation (2) is also of importance in the design of microelectronic devices aimed to sense data in the realm of Radio Frequency Identification Devices (RFID's). Here, the

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performance of the device in its sensing (receiving) state is related to its transmitting properties.

II. The Antenna in its Transmitting Mode

In its transmitting mode, the antenna is, at its accessible ports, excited by either prescribed voltages $\left[\hat{V}^T\right]$ or prescribed electric currents $\left[\hat{I}^T\right]$. In view of the passivity of the medium in $\mathcal{D}^\infty$ and the property that $\hat{\boldsymbol{E}}^T$ and $\hat{\boldsymbol{H}}^T$ consist of outgoing waves, the linear relationship between $\left[\hat{V}^T\right]$ and $\left[\hat{I}^T\right]$ can be expressed either as

$$\left[\hat{V}^T\right] = \left[\hat{Z}^T\right]\left[\hat{I}^T\right], \tag{4}$$

where $\left[\hat{Z}^T\right]$ is the *radiation impedance* of the antenna or as

$$\left[\hat{I}^T\right] = \left[\hat{Y}^T\right]\left[\hat{V}^T\right], \tag{5}$$

where $\left[\hat{Y}^T\right]$ is the *radiation admittance* of the antenna. The non-vanishing of the radiated power of the antenna in any transmitting state entails that both $\left[\hat{Z}^T\right]$ and $\left[\hat{Y}^T\right]$ are dissipative. The uniqueness of the EM field problem in the transmitting state further ensures that $\left[\hat{Z}^T\right]$ and $\left[\hat{Y}^T\right]$ are each other's inverse.

In the transmitting mode, the field distributions on $\partial\mathcal{D}$ are linearly related to their Kirchhoff-port excitation quantities. For the case of excitation via prescribed voltages $\hat{V}_n^T$, we write

$$\{\hat{\boldsymbol{E}}^T, \hat{\boldsymbol{H}}^T\} = \sum_{n=1}^{N}\{\hat{\mathbf{e}}_n^{T;V}, \hat{\boldsymbol{h}}_n^{T;V}\}\hat{V}_n^T. \tag{6}$$

Substitution of (6) in the right-hand side of (5) yields for this case

$$\int_{\partial\mathcal{D}}\left(\hat{\boldsymbol{E}}^T\times\hat{\boldsymbol{H}}^i - \hat{\boldsymbol{E}}^i\times\hat{\boldsymbol{H}}^T\right)\cdot\boldsymbol{\nu}\,dA = \sum_{n=1}^{N}\hat{I}_n^G\hat{V}_n^T, \tag{7}$$

in which

$$\hat{I}_n^G = \int_{\partial\mathcal{D}}\left(\hat{\mathbf{e}}_n^{T;V}\times\hat{\boldsymbol{H}}^i - \hat{\boldsymbol{E}}^i\times\hat{\boldsymbol{h}}_n^{T;V}\right)\cdot\boldsymbol{\nu}\,dA \quad \text{for } n = 1,\ldots,N. \tag{8}$$

For the case of excitation via prescribed electric currents $\hat{I}_n^T$, we write

$$\{\hat{\boldsymbol{E}}^T, \hat{\boldsymbol{H}}^T\} = \sum_{n=1}^{N}\{\hat{\mathbf{e}}_n^{T;I}, \hat{\boldsymbol{h}}_n^{T;I}\}\hat{I}_n^T. \tag{9}$$

Substitution of (9) in the right-hand side of (5) yields for this case

$$\int_{\partial\mathcal{D}}\left(\hat{\boldsymbol{E}}^T\times\hat{\boldsymbol{H}}^i - \hat{\boldsymbol{E}}^i\times\hat{\boldsymbol{H}}^T\right)\cdot\boldsymbol{\nu}\,dA = \sum_{n=1}^{N}\hat{V}_n^G\hat{I}_n^T, \tag{10}$$

in which

$$\hat{V}_n^G = \int_{\partial\mathcal{D}}\left(\hat{\mathbf{e}}_n^{T;I}\times\hat{\boldsymbol{H}}^i - \hat{\boldsymbol{E}}^i\times\hat{\boldsymbol{h}}_n^{T;I}\right)\cdot\boldsymbol{\nu}\,dA \quad \text{for } n = 1,\ldots,N. \tag{11}$$

III. The Antenna in its Receiving Mode

In its receiving mode, $\left[\hat{V}^R\right]$ and $\left[\hat{I}^R\right]$ are linearly related via the *loading conditions* as observed at the accessible ports,

$$\left[\hat{V}^R\right] = -\left[\hat{Z}^L\right]\left[\hat{I}^R\right], \tag{12}$$

in which $\left[\hat{Z}^L\right]$ is the *load impedance*, or

$$-\left[\hat{I}^R\right] = \left[\hat{Y}^L\right]\left[\hat{V}^R\right], \tag{13}$$

in which $\left[\hat{Y}^L\right]$ is the *load admittance*. The passivity of the configuration in the receiving state ensures that both $\left[\hat{Z}^L\right]$ and $\left[\hat{Y}^L\right]$ are passive, while in view, again, of the uniqueness of the field scattering (+ absorption) problem, $\left[\hat{Z}^L\right]$ and $\left[\hat{Y}^L\right]$ are each other's inverse. The $-$sign in (12) and (13) reflects that the electric currents are oriented into $\mathcal{D}$.

IV. The Equivalent Thévenin Circuit

Substituting (6) and (10) in (2) and observing that the resulting relation has to hold for any choice of $\{\hat{I}_n^T\}$ we arrive at the relation

$$\left[\hat{V}^R\right] - \left[\hat{Z}^T\right]\left[\hat{I}^R\right] = \left[\hat{V}^G\right]. \tag{14}$$

Equation (14) is representative for a *voltage source* circuit, i.e., a Thévenin circuit, with $\left[\hat{Z}^T\right]$ as *internal impedance* and $\left[\hat{V}^G\right]$ as *excitation*. Combined with (12) and (14) yields

$$\left[\hat{I}^R\right] = -\left[\hat{Z}^T + \hat{Z}^L\right]^{-1}\left[\hat{V}^G\right]. \tag{15}$$

V. The Equivalent Norton Circuit

Substituting (7) and (9) in (2) and observing that the resulting relation has to hold for any choice of $\{\hat{V}_n^T\}$ we arrive at the relation

$$\left[\hat{I}^R\right] - \left[\hat{Y}^T\right]\left[\hat{V}^R\right] = \left[\hat{I}^G\right]. \tag{16}$$

Equation (16) is representative for a *electric-current source* circuit, i.e., a Norton circuit, with $\left[\hat{Y}^T\right]$ as *internal admittance* and $\left[\hat{I}^G\right]$ as *excitation*. Combined with (13) and (16) yields

$$\left[\hat{V}^R\right] = -\left[\hat{Y}^T + \hat{Y}^L\right]^{-1}\left[\hat{I}^G\right]. \tag{17}$$

VI. Equivalence of the Two Circuits

In view of the uniqueness of the scattering and absorption problem in the receiving state, the descriptions of the antenna's receiving properties via the equivalent Thévenin and Norton circuits should be completely equivalent. To investigate the consequences of this, it is observed that (14) can be rewritten as

$$\left[\hat{I}^R\right] - \left[\hat{Y}^T\right]\left[\hat{V}^R\right] = \left[\hat{Y}^T\right]\left[\hat{V}^G\right]. \tag{18}$$

Equation (18) is equivalent to (16) provided that

$$\left[\hat{I}^G\right] = \left[\hat{Y}^T\right]\left[\hat{V}^G\right]. \tag{19}$$

Similarly, Equation (16) is equivalent to (14) provided that

$$\left[\hat{V}^G\right] = \left[\hat{Z}^T\right]\left[\hat{I}^G\right]. \tag{20}$$

The interrelations (19) and (20) between the exciting quantities in the two circuits represent an interesting EM field property of the right-hand sides in (8) and (11). This interrelation seems to have been unnoticed in the antenna literature.

VII. Conclusions

The complete equivalence of the Thévenin (voltage source) and the Norton (electric-current source) circuits that describe the receiving properties of an antenna, that has been a source under doubt in [1]–[7], has been proven. The basic ingredients in the proof are: the Lorentz reciprocity relation of the time-convolution type and the uniqueness of the EM field in the transmitting as well as in the receiving mode of the antenna. Since the voltages and the electric currents in the two representations are the same, also the properties derived from them, such as the energy dissipated in the load, are the same.

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